

Yang-Baxter σ model: Quantum aspects

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Abstract

We study the quantum properties at one-loop of the Yang-Baxter σ -models introduced by Klimčík[1, 2]. The proof of the one-loop renormalizability is given, the one-loop renormalization flow is investigated and the quantum equivalence is studied.

1 Introduction

The Yang-Baxter σ -models were first introduced by Klimčík [1, 2] as a special case, at the classical level, of a non-linear σ -model with Poisson-Lie symmetry[3, 4]. Recall that the Poisson-Lie symmetry appears to be the natural generalization of the so-called Abelian T -duality[5] and non-Abelian T -duality [6, 7, 8] of non-linear σ -models. In particular, two dynamically equivalent σ -models can be obtained at the classical level providing that Poisson-Lie symmetry condition holds. That condition takes a very elegant formulation in the case where the target space is a compact semi-simple Lie group which naturally leads to the concept of the Drinfeld double[9]. The Drinfeld double is the $2n$ -dimensional linear space where both dynamically equivalent theories live. For the Poisson-Lie σ -models, a proof of the one-loop renormalizability and quantum equivalence was given in [10, 11, 12, 13]. We are interested by a special class of classical Poisson-Lie σ -models, the Yang-Baxter σ -models. Those classical models exhibit the special feature to be both Poisson-Lie symmetric with respect to the right action of the group on itself and left invariant. Thus, using the right Poisson-Lie symmetry or the left group action leads to two different dual theories. Those two dynamically equivalent dual pair of models live in two non-isomorphic Drinfeld doubles, the cotangent bundle of the Lie group for the left action, and the complexified of the Lie group for the right Poisson-Lie symmetry. Classical properties were investigated in the past and it has been showed that Yang-Baxter σ -models are integrable [1]. More recently, based on the previous work of Refs.[14, 15, 16], authors of Ref.[17] proved that they belong to a more general class of integrable σ -models. In particular, they showed that the ε -deformation parameter of the Poisson-Lie symmetry can be re-interpreted as a classical q -deformation of the Poisson-Hopf algebra.

If classical properties are well investigated, very little is known about the quantum version of the Yang-Baxter σ -models. In the case where the Lie group is $SU(2)$, the Yang-Baxter σ -model coincides with the anisotropic principal model which is known to be one-loop renormalizable. This low dimensional result can let us hope a generalization for any Yang-Baxter σ -models. However, contrary to the anisotropic principal model, the Yang-Baxter σ -models contain a non-vanishing torsion which could potentially gives rise to some difficulties. On the other hand, another generalization of the anisotropic chiral model, the squashed group models are one-loop renormalizable for a special choice of torsion [22].

Furthermore, the one-loop renormalizability of the Poisson-Lie σ -model cannot provide any help here since the proof was established for a theory containing n^2 parameters when the Yang-Baxter σ -models contain only two: the ε deformation and the coupling constant t . At the quantum level, the Yang-Baxter σ -models are no more a special case of the Poisson-Lie σ models. The

main result of this article consists in proving the one-loop renormalizability of Yang-Baxter σ -models.

The plan of the article is as follows. In Section 2 we introduced the Yang-Baxter σ -models on a Lie group and all algebraic tools needed. In section 3, the counter-term of the Yang-Baxter σ -models, i.e. the Ricci tensor, is calculated. Section 4 is dedicated to the proof of the one-loop renormalizability, and the computation of the renormalization flow is done in Section 5. In Section 6, we study the quantum equivalence and we express the Yang-Baxter σ -action in terms of the usual one of the Poisson-Lie σ -models. Outlooks take place in Section 7.

2 Yang-Baxter σ -models

2.1 The complexified double

We considered the case of the Yang-Baxter models studied in Ref.[1]. In that case the Drinfeld double D can be the complexification of a simple compact and simply-connected Lie group G , i.e. $D = G^{\mathbb{C}}$, or the cotangent bundle T^*G . Let us consider the case of the complexified Drinfeld double, it turns out that $D = G^{\mathbb{C}}$ admits the so-called Iwasawa decomposition

$$G^{\mathbb{C}} = GAN = ANG. \quad (1)$$

In particular, if $D = SL(n, \mathbb{C})$, then the group AN can be identified with the group of upper triangular matrices of determinant 1 and with positive numbers on the diagonal and $G = SU(n)$.

Furthermore, the Lie algebra \mathcal{D} turns out to be the complex Lie algebra $\mathcal{G}^{\mathbb{C}}$, which suggests to use the roots space decomposition of $\mathcal{G}^{\mathbb{C}}$:

$$\mathcal{G}^{\mathbb{C}} = \mathcal{H}^{\mathbb{C}} \bigoplus (\bigoplus_{\alpha \in \Delta} \mathbb{C} E_{\alpha}), \quad (2)$$

where Δ is the space of all roots. Consider the Killing-Cartan form κ on $\mathcal{G}^{\mathbb{C}}$, and let us take an orthonormal basis H_i in the r -dimensional Cartan sub-algebra $\mathcal{H}^{\mathbb{C}}$ of $\mathcal{G}^{\mathbb{C}}$ with respect to the bilinear form κ on $\mathcal{G}^{\mathbb{C}}$, i.e:

$$\kappa(H_i, H_j) \equiv \delta_{ij} \quad (3)$$

This permits to define a canonical bilinear form on \mathcal{H}^* , and more specifically endows the roots space $\Delta \subset \mathcal{H}^*$ with an Euclidean metric, i.e.

$$(\alpha, \beta) = \delta^{ij} \alpha_i \beta_j, \quad \alpha_i = \alpha(H_i).$$

Moreover, the inner product on the roots space part of $\mathcal{G}^{\mathbb{C}}$ is chosen such as:

$$\kappa(E_{\alpha}, E_{-\alpha}) \equiv 1, \quad (4)$$

and to fix the normalization, we impose the following non-linear condition $E_\alpha = E_{-\alpha}^\dagger$. With all those conventions, the generators of $\mathcal{G}^\mathbb{C}$ verify:

$$\begin{aligned} [H_i, E_\alpha] &= \alpha_i E_\alpha & [E_\alpha, E_{-\alpha}] &= \alpha^i H_i \\ [H_i, H_j] &= 0 & [E_\alpha, E_\beta] &= N_{\alpha, \beta} E_{\alpha+\beta}, \quad \alpha + \beta \in \Delta. \end{aligned} \quad (5)$$

The structure constants $N_{\alpha, \beta}$ vanish if $\alpha + \beta$ is not a root.

Since $\mathcal{G}^\mathbb{C}$ is a Lie algebra, the structure constants verify the Jacobi identity which leads on one hand to:

$$N_{\alpha, \beta} = N_{\beta, -\alpha-\beta} = N_{-\alpha-\beta, \alpha}, \quad (6)$$

and on the other hand to:

$$N_{\alpha, \beta+(k-1)\alpha} N_{\beta+k\alpha, -\alpha} + N_{-\alpha, \beta+(k+1)\alpha} N_{\alpha, \beta+k\alpha} = -(\alpha, \beta + k\alpha). \quad (7)$$

In the non-vanishing case, the structure constants $N_{\alpha, \beta}$ can be calculated from the last relation

$$N_{\alpha, \beta}^2 = n(m+1)(\alpha, \alpha), \quad (8)$$

with $(n, m) \in \mathbb{N}$ such that $\beta + n\alpha$ and $\beta - m\alpha$ are the last roots of the chain containing β (see Ref.[21] for more details).

Since H_i is an orthonormal basis in $\mathcal{H}^\mathbb{C}$, we obtain the relations:

$$\sum_{\alpha \in \Delta} \alpha_i \alpha_j = \delta_{ij}, \quad \text{and} \quad \sum_{\alpha \in \Delta} (\alpha, \alpha) = r. \quad (9)$$

A basis of the compact Lie real form \mathcal{G} of $\mathcal{G}^\mathbb{C}$ can be obtained by the following transformations:

$$T_i = iH_i, \quad B_\alpha = \frac{i}{\sqrt{2}}(E_\alpha + E_{-\alpha}) \quad C_{\bar{\alpha}} = \frac{1}{\sqrt{2}}(E_\alpha - E_{-\alpha}), \quad (10)$$

with $\alpha \in \Delta^+$ (positive roots). With our choice of normalization, the vectors of the basis verify $\kappa(T_i, T_j) \equiv \kappa_{ij} = -\delta_{ij}$, $\kappa(B_\alpha, B_\beta) \equiv \kappa_{\alpha\beta} = -\delta_{\alpha\beta}$, $\kappa(C_{\bar{\alpha}}, C_{\bar{\beta}}) \equiv \kappa_{\bar{\alpha}\bar{\beta}} = -\delta_{\alpha\beta}$ and all others are zero.

Let us define now a \mathbb{R} -linear operator $R : \mathcal{G} \rightarrow \mathcal{G}$ such that:

$$RT_i = 0, \quad RB_\alpha = C_{\bar{\alpha}}, \quad RC_{\bar{\alpha}} = -B_\alpha, \quad (11)$$

this operator R is the so-called the Yang-Baxter operator [2] which satisfies the following modified Yang-Baxter equation:

$$[RA, RB] = R([RA, B] + [A, RB]) + [A, B], \quad (A, B) \in \mathcal{G}. \quad (12)$$

Let us define the skew-symmetric bracket:

$$[A, B]_R = [RA, B] + [A, RB], \quad (A, B) \in \mathcal{G}, \quad (13)$$

which fulfills the Jacobi identity, and defines a new Lie algebra $(\mathcal{G}, [\cdot, \cdot]_R)$. It turns out that this new algebra is nothing but the Lie algebra of the AN group of the Iwasawa decomposition of $G^\mathbb{C}$ and will be denoted $\mathcal{G}_\mathbb{R}$ the dual algebra.

2.2 The Yang-Baxter action

We shall now consider the action of the Yang-Baxter σ -models[2] expressed on the Lie group G , which takes the expression:

$$\mathcal{S}(g) = -\frac{1}{2t} \int \kappa(g^{-1}\partial_+g, (\mathbb{1} - \varepsilon R)^{-1}g^{-1}\partial_-g) d\xi^+ d\xi^-, \quad g \in G \quad (14)$$

where $\partial_+ = \partial_\tau + \partial_\sigma$ and $\partial_- = \partial_\tau - \partial_\sigma$, $\mathbb{1}$ is the identity map on \mathcal{G} , t is the *coupling constant* and ε is the *deformation* parameter.

We can immediately check that the Yang-Baxter models (14) are left action invariant, G acting on himself. Concerning the right Poisson-Lie symmetry, it is well known that such σ -models have to fulfill a zero curvature condition to be Poisson-Lie invariant. Indeed, if we take the following \mathcal{G}^* -valued Noether current 1-form $J(g)$:

$$J(g) = -(\mathbb{1} + \varepsilon R)^{-1}g^{-1}\partial_+g d\xi^+ + (\mathbb{1} - \varepsilon R)^{-1}g^{-1}\partial_-g d\xi^-, \quad (15)$$

we can easily verify that the fields equations of (14) are equivalent to the following zero curvature condition:

$$\partial_+ J_-(g) - \partial_- J_+(g) + \varepsilon [J_-(g), J_+(g)]_R = 0. \quad (16)$$

We remark that if the deformation ε vanishes then the action of the group G is an isometry, since the Noether current are closed 1-forms on the world-sheets and the action (14) coincides with that of the principal chiral σ -model. The operator $(\mathbb{1} - \varepsilon R)^{-1}$ on \mathcal{G} can be decomposed in a symmetric part interpreted as a metric g on G and a skew-symmetric part interpreted as a torsion potential h on G . An attentive study of the action (14) gives the following expressions for g and h :

$$g = \kappa_{ij}(g^{-1}dg)^i(g^{-1}dg)^j + \frac{1}{1 + \varepsilon^2} \left(\kappa_{\alpha\beta}(g^{-1}dg)^\alpha(g^{-1}dg)^\beta + \kappa_{\bar{\alpha}\bar{\beta}}(g^{-1}dg)^{\bar{\alpha}}(g^{-1}dg)^{\bar{\beta}} \right), \quad (17)$$

$$h = -\frac{\varepsilon}{1 + \varepsilon^2} (g^{-1}dg)^\alpha \wedge (g^{-1}dg)^{\bar{\alpha}} \kappa_{\alpha\bar{\alpha}}. \quad (18)$$

In order to prove the one-loop renormalizability, we need to calculate the Ricci tensor associated to the manifold (G, g, h) .

3 Counter-term of the Yang-Baxter σ -models

In this paper, for the calculus of the counter-term, we choose the standard approach[18] based on the Ricci tensor. This choice provides a clear and an elegant expression of the counter-term in terms of the roots of $G^\mathbb{C}$. However the calculus could have been done by using our formula of [12] for the counter-term in an equivalent way.

3.1 Geometry with torsion on a Lie group G

Let us consider a pseudo-Riemannian manifold (G, g) as the base of its frame bundle, where G is a compact semi-simple Lie group and g a non-degenerated metric. Moreover, we choose the left Maurer-Cartan form $g^{-1}dg$, $g \in G$ as the basis of 1-forms on G , and in that basis the metric coefficients g_{ab} and the torsion components T_{abc} are all constant.

On that frame bundle we define a metric connection Ω with its covariant derivative D such that $Dg = 0$. Furthermore, if we define by d^D the exterior covariant derivative, the torsion can be written $T = d^D(g^{-1}dg)$. From these definitions we will obtain the expression of the connection Ω .

Metric connection.

By using the relation $Dg = 0$ we obtain:

$$\Omega^s_{ac}g_{sb} + \Omega^s_{bc}g_{as} = 0. \quad (19)$$

With g_{ab} constant and if we denote $\Omega_{abc} = g_{as}\Omega^s_{bc}$, the previous relation becomes:

$$\Omega_{abc} = -\Omega_{bac} \quad (20)$$

Thus the two first indices of the connection Ω are skew-symmetric.

The torsion.

We said that the torsion verifies $T = d^D(g^{-1}dg)$ or in terms of components:

$$T^a = \Omega^a_b \wedge (g^{-1}dg)^b + d(g^{-1}dg)^a. \quad (21)$$

Since $g^{-1}dg$ is the left Maurer-Cartan form on \mathcal{G} , we get:

$$d(g^{-1}dg) = -(g^{-1}dg) \wedge (g^{-1}dg),$$

on the other hand T^a is the 2-form torsion, i.e. we can write it as:

$$T^a = \frac{1}{2}T^a_{bc}(g^{-1}dg)^b \wedge (g^{-1}dg)^c.$$

Consequently, the components of the torsion are related to the skew-symmetric part of the connection as:

$$T^a_{bc} = \Omega^a_{cb} - \Omega^a_{bc} - f_{bc}^a, \quad (22)$$

with f_{bc}^a the structure constants of the Lie algebra \mathcal{G} .

Note that in the case of the non-linear σ -models the torsion is defined by $T = dh$ where h is the 2-form potential torsion, we will exploit that a little further to express the connection for the Yang-Baxter σ -models.

The connection.

From the relations (20)(22), we can find the components of the connection:

$$2\Omega_{abc} = (-T_{abc} - T_{bca} + T_{cab}) + (f_{abc} - f_{cab} - f_{bca}), \quad (23)$$

with the conventions $\Omega_{abc} = g_{as}\Omega_{bc}^s$, $T_{abc} = g_{as}T_{bc}^s$ and $f_{bca} = f_{bc}^s g_{sa}$. Let us introduce the Levi-Civita connection L which is in fact the second term of the r.h.s in Eq.(23), and rewrite the connection Ω for a totally skew-symmetric torsion:

$$\Omega_{abc} = L_{abc} - \frac{1}{2}T_{abc}. \quad (24)$$

The curvature and the Ricci.

By definition the 2-form curvature F fulfills $F = d^D\Omega$, i.e.

$$F_{ab}^a = d\Omega_b^a + \Omega_s^a \wedge \Omega_b^s. \quad (25)$$

Moreover, since Ω_b^a is a 1-form of G , $\Omega_b^a = \Omega_{bc}^a(g^{-1}dg)^c$, we obtain the general expression for the curvature:

$$F_{bcd}^a = \Omega_{sc}^a \Omega_{bd}^s - \Omega_{sd}^a \Omega_{bc}^s - \Omega_{bs}^a f_{cd}^s. \quad (26)$$

The Ricci tensor is such that $Ric_{ab} = F_{asb}^s$ and can be written as:

$$Ric_{ab} = -\Omega_{ar}^s (\Omega_{bs}^r + f_{bs}^r). \quad (27)$$

We are now able to decompose the symmetric and skew-symmetric parts of the Ricci tensor in terms of the torsion-less Ricci tensor Ric^L and the torsion T as:

$$Ric_{(ab)} = Ric_{(ab)}^L + \frac{1}{4}T_{as}^r T_{br}^s \quad (28)$$

$$Ric_{[ab]} = \frac{1}{2}f_{at}^s T_{bs}^t - \frac{1}{2}g_{at}f_{sr}^t g^{ru}T_{bu}^s + \frac{1}{2}g_{st}f_{ar}^t g^{ru}T_{bu}^s - (a \leftrightarrow b). \quad (29)$$

3.2 Application to Yang-Baxter

Ricci symmetric part :

Recall that in the case of the Yang-Baxter σ -models and with our normalization choice, the metric is given by:

$$g_{ij} = -\delta_{ij}, \quad g_{\alpha\beta} = -\frac{1}{1+\varepsilon^2}\delta_{\alpha\beta}, \quad g_{\bar{\alpha}\bar{\beta}} = -\frac{1}{1+\varepsilon^2}\delta_{\alpha\beta}. \quad (30)$$

Let us introduce the bi-invariant connection Γ on the Lie group G , it corresponds to the Levi-Civita connection in the case of a vanishing deformation, i.e. $\Gamma = L(\varepsilon = 0)$. From the equations (23) we can obtain the Levi-Civita coefficients:

$$L_{\bar{\alpha}i}^\alpha = -L_{\alpha i}^{\bar{\alpha}} = (1 - \varepsilon^2)\Gamma_{\bar{\alpha}i}^\alpha \quad (31)$$

$$L_{i\bar{\alpha}}^\alpha = -L_{i\alpha}^{\bar{\alpha}} = (1 + \varepsilon^2)\Gamma_{i\bar{\alpha}}^\alpha, \quad (32)$$

where we keep the convention for the indices $i \in \mathcal{H}$ and $\alpha \in \Delta^+$. All others Levi-Civita coefficients are equal to those of the bi-invariant connection Γ . We can now express the torsion-less Ricci tensor Ric^L as a deformation of the usual Ricci tensor Ric^Γ of the bi-invariant connection on Lie group, i.e.

$$Ric^L_{\alpha\beta} = Ric^\Gamma_{\alpha\beta} - \frac{\varepsilon^2}{2}(\alpha, \alpha)\delta_{\alpha\beta} \quad (33)$$

$$Ric^L_{\bar{\alpha}\bar{\beta}} = Ric^\Gamma_{\bar{\alpha}\bar{\beta}} - \frac{\varepsilon^2}{2}(\alpha, \alpha)\delta_{\alpha\beta} \quad (34)$$

$$Ric^L_{ij} = (1 + \varepsilon^2)^2 Ric^\Gamma_{ij} \quad (35)$$

It is well-known that for the Riemannian bi-invariant structure the Ricci tensor takes the expression:

$$Ric^\Gamma_{ab} = -\frac{1}{4}\kappa_{ab}, \quad (a, b) \in G, \quad (36)$$

therefore, the components of Ric^L are the following:

$$Ric^L_{\alpha\beta} = Ric^L_{\bar{\alpha}\bar{\beta}} = -\frac{1}{4}\kappa_{\alpha\beta} - \frac{\varepsilon^2}{2}(\alpha, \alpha)\delta_{\alpha\beta} \quad (37)$$

$$Ric^L_{ij} = -\frac{1}{4}\kappa_{ij}(1 + \varepsilon^2)^2 \quad (38)$$

Concerning the contribution of the Torsion to the symmetric part of the Ricci tensor, we have to express the Torsion in terms of the constant structures of G . For a non-linear σ -model the Torsion 3-form is calculated from the potential torsion 2-form such $T = dh$, which implies that:

$$T_{abc} = -3f_{[ab}^s h_{c]s}, \quad (a, b, c, s) \in G. \quad (39)$$

Moreover, since the torsion potential involves only root indices

$$h = -\frac{\varepsilon}{1 + \varepsilon^2}(g^{-1}dg)^\alpha \wedge (g^{-1}dg)^{\bar{\alpha}}\kappa_{\alpha\alpha},$$

the torsion components vanish for the Cartan sub-algebra indices ($T_{ibc} = 0$). We can now calculate the torsion contribution, and we obtain for the non-vanishing coefficients:

$$\frac{1}{4}T^r_{\alpha s}T^s_{\alpha r} = \frac{1}{4}T^r_{\bar{\alpha} s}T^s_{\bar{\alpha} r} = \frac{\varepsilon^2}{2}\left(\frac{1}{2}\kappa_{\alpha\alpha} + (\alpha, \alpha)\right). \quad (40)$$

In the calculus we used the fact that the Killing κ can be expressed in terms of the root α and the constant structures $N_{\alpha, \beta}$ such as:

$$-\frac{1}{2}\kappa_{\alpha\alpha} = \alpha^i \alpha_i + \frac{1}{2} \sum_{\beta \in \Delta^+} (N_{\alpha, \beta})^2 + (N_{\alpha, -\beta})^2. \quad (41)$$

Adding both contributions to the Ricci tensor and using our normalization, we obtain the final expression of the symmetric part:

$$Ric_{\alpha\beta} = Ric_{\bar{\alpha}\bar{\beta}} = -\frac{\kappa_{\alpha\beta}}{4}(1 - \varepsilon^2) = \frac{1}{4}(1 - \varepsilon^2)\delta_{\alpha\beta} \quad (42)$$

$$Ric_{ij} = -\frac{\kappa_{ij}}{4}(1 + \varepsilon^2)^2 = \frac{\delta_{ij}}{4}(1 + \varepsilon^2)^2. \quad (43)$$

We observe that, in the case of the Yang-Baxter model, the torsion induced by the Poisson-Lie symmetry is precisely that which avoids the dependence of the Ricci tensor in the root length (α, α) .

Ricci skew-symmetric part :

Using the fact the $T_{iab} = 0$, the only non-vanishing non-diagonal components of the Ricci tensor can be written:

$$Ric_{\alpha\bar{\alpha}} = 2f_{\alpha\bar{\beta}\gamma}T_{\bar{\alpha}\bar{\beta}\gamma}\kappa^{\gamma\gamma}\kappa^{\bar{\beta}\bar{\beta}} - f_{\bar{\alpha}\beta\gamma}T_{\alpha\beta\gamma}\kappa^{\gamma\gamma}\kappa^{\beta\beta} - f_{\bar{\alpha}\bar{\beta}\bar{\gamma}}T_{\alpha\bar{\beta}\bar{\gamma}}\kappa^{\bar{\gamma}\bar{\gamma}}\kappa^{\bar{\beta}\bar{\beta}}. \quad (44)$$

The first r.h.s term can be expressed as a function of the structure constants, $N_{\alpha,\beta}$ such as:

$$2f_{\alpha\bar{\beta}\gamma}T_{\bar{\alpha}\bar{\beta}\gamma}\kappa^{\gamma\gamma}\kappa^{\bar{\beta}\bar{\beta}} = 2\varepsilon \sum_{\beta \in \Delta^+} (N_{\alpha,\beta})^2 - (N_{\alpha,-\beta})^2. \quad (45)$$

The two other terms are nothing but the contribution of the roots space (see Eq.(41)) to the component $\kappa_{\bar{\alpha}\bar{\alpha}}$ of the Killing form, i.e.:

$$-f_{\bar{\alpha}\beta\gamma}T_{\alpha\beta\gamma}\kappa^{\gamma\gamma}\kappa^{\beta\beta} - f_{\bar{\alpha}\bar{\beta}\bar{\gamma}}T_{\alpha\bar{\beta}\bar{\gamma}}\kappa^{\bar{\gamma}\bar{\gamma}}\kappa^{\bar{\beta}\bar{\beta}} = -\frac{\varepsilon}{2}\left(\kappa_{\bar{\alpha}\bar{\alpha}} + 2(\alpha, \alpha)\right). \quad (46)$$

By summing the Bianchi relations (7) on positive roots, we obtain that:

$$\sum_{\beta \in \Delta^+} (N_{\alpha,\beta})^2 - (N_{\alpha,-\beta})^2 = -2(\rho, \alpha) + (\alpha, \alpha), \quad (47)$$

with

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$$

the Weyl vector.

Finally, the skew-symmetric part of the Ricci tensor is given by:

$$Ric_{\alpha\bar{\alpha}} = -Ric_{\bar{\alpha}\alpha} = -\varepsilon\left(2(\alpha, \rho) + \frac{1}{2}\kappa_{\alpha\alpha}\right). \quad (48)$$

4 One-loop renormalizability

At one-loop the counter-terms for a non-linear σ -model[18] on G are given by:

$$\frac{1}{4\pi\epsilon} \int Ric_{ab}(g^{-1}\partial_-g)^a(g^{-1}\partial_+g)^b, \quad \epsilon = 2 - d. \quad (49)$$

We require, for the renormalizability, that all divergences have to be absorbed by fields-independent deformations of the parameters (t, ϵ) and a possible non-linear fields renormalization of the fields $(g^{-1}\partial_{\pm}g)^a$. Thus, if we suppose that all parameters are the independent coupling constants of the theory, the Ricci tensor in our frame has to verify the relations:

$$Ric_{ab} = -\chi_0(\mathbb{1} - \epsilon R)_{ab}^{-1} + \chi_\epsilon \frac{\partial}{\partial \epsilon}(\mathbb{1} - \epsilon R)_{ab}^{-1} + D_b u_a, \quad (50)$$

with u a vector that contributes to the fields renormalization, χ_0 and χ_ϵ are coordinates-independent. Decomposing into symmetric and skew-symmetric parts, the previous relation for the Yang-Baxter σ -models becomes:

$$Ric_{ij} = -\chi_0 g_{ij} \quad (51)$$

$$Ric_{\alpha\alpha} = -\chi_0 g_{\alpha\alpha} - \chi_\epsilon \frac{2\epsilon}{1 + \epsilon^2} g_{\alpha\alpha} \quad (52)$$

$$Ric_{\alpha\bar{\alpha}} = -\chi_0 h_{\alpha\bar{\alpha}} + \chi_\epsilon \frac{1 - \epsilon^2}{\epsilon(1 + \epsilon^2)} h_{\alpha\bar{\alpha}} + D_{\bar{\alpha}} u_\alpha. \quad (53)$$

From the equations (51) and (52), we extract immediately:

$$\chi_0 = \frac{1}{4}(1 + \epsilon^2)^2, \text{ and } \chi_\epsilon = -\frac{1}{4}\epsilon(1 + \epsilon^2)^2. \quad (54)$$

Since χ_0 and χ_ϵ are now fixed, they have to fulfill in the same time the relation (53), which gives the following constraint:

$$\epsilon \left(-\frac{1}{2} + 2(\rho, \alpha) \right) = -\frac{1}{2}\epsilon + D_{\bar{\alpha}} u_\alpha. \quad (55)$$

Furthermore, the covariant derivative of u can be easily calculated:

$$Du = -\frac{1}{2} \sum_{\alpha \in \Delta^+} (u, \alpha) (g^{-1}dg)^\alpha \wedge (g^{-1}dg)^{\bar{\alpha}}. \quad (56)$$

Let us define the vector $\epsilon \bar{u} = u$, and insert (56) in the constraint (55) we obtain:

$$(4\rho - \bar{u}, \alpha) = 0. \quad (57)$$

Then, if we impose $\bar{u} = 4\rho$ the constraint is fulfilled for any root α since $(., .)$ is the canonical scalar product on \mathbb{R}^r . We can conclude that the Yang-Baxter σ -models are one-loop renormalizable.

We note that it is quite elegant to find a field renormalization given by the Weyl vector.

5 Renormalization flow

Let us introduce the β -functions of the two parameters (t, ε) , they satisfy:

$$\beta_t = \frac{dt}{d\lambda} = -t^2 \chi_0, \quad \beta_\varepsilon = \frac{d\varepsilon}{d\lambda} = t \chi_\varepsilon, \quad (58)$$

where $\lambda = \frac{1}{\pi} \ln \mu$, with μ the mass energy scale. We obtain the following system of differential equations:

$$\frac{dt}{d\lambda} + \frac{1}{4}(1 + \varepsilon^2)^2 t^2 = 0 \quad (59)$$

$$\frac{d\varepsilon}{d\lambda} + \frac{1}{4}\varepsilon(1 + \varepsilon^2)^2 t = 0. \quad (60)$$

The set of differential equations can be exactly solved, and solutions take the following general expressions:

$$t(\varepsilon) = A\varepsilon, \quad \hat{\lambda}(\varepsilon) = B\lambda(\varepsilon) = \frac{3}{2} \arctan \varepsilon + \frac{1 + \frac{3}{2}\varepsilon^2}{\varepsilon(1 + \varepsilon^2)^2} + C, \quad (61)$$

with $(A, B, C) \in \mathbb{R}$ three integrative constants. We note that divergences occur for ε and t when the energy scale $\hat{\lambda}$ goes to $\pm \frac{3\pi}{4} + C$. On the other hand, for $\hat{\lambda} \rightarrow \infty$ the parameters ε and t are vanishing, leading to an asymptotic freedom. We can illustrate the situation with the following plot (Fig.1) of λ as a function of ε where we choose $B = 1$ and $C = 0$.

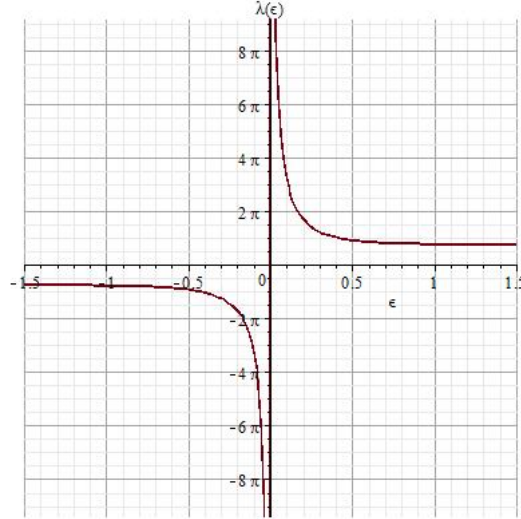


Figure 1: Energy scale λ as a function of the deformation parameter ε

6 Poisson-Lie models and duality

Now we will express the Yang-Baxter σ -models in terms of the usual Poisson-Lie σ -models' expression. Recall that general *Right* symmetric Poisson-Lie σ -models can be written:

$$\mathcal{S}(g) = \frac{1}{2t} \int (\partial_- g g^{-1})^a (M + \Pi_R(g))_{ab}^{-1} (\partial_+ g g^{-1})^b. \quad (62)$$

Here $\Pi_R(g)$ is the so-called *Right* Poisson-Lie bi-vector and M an n^2 real matrix.

Using the adjoint action of an element $g \in G$ we can rewrite the action (14) such as the previous (62), with

$$\Pi_R(g) = \text{Ad}_g R \text{Ad}_{g^{-1}} - R \text{ and } M = \frac{1}{\varepsilon} \mathbb{1} - R.$$

Let us focus on the dual models, as evoked earlier there exists two non-isomorphic Drinfeld doubles for the action (62). Consequently, we have two different dual theories for one single initial theory on G , and all three are classically equivalent. We will consider each case and argue that they are all quantum-equivalent at one-loop.

We start by considering the Drinfeld double $D = G^{\mathbb{C}}$, in that case we saw that the dual group is the factor AN in the Iwasawa decomposition. The corresponding algebra is the Lie algebra $\mathcal{G}_{\mathbb{R}}$ generated by the \mathbb{R} -linear operator $(R - i)$ on \mathcal{G} , whose its group is a non-compact real form of $G^{\mathbb{C}}$ (see [2, 20] for details). The dual action can be expressed as:

$$\mathcal{S}(\hat{g}) = \frac{1}{2\varepsilon t} \int (\partial_- \hat{g} \hat{g}^{-1})_a [(M^{-1} + \hat{\Pi}_R(\hat{g}))^{-1}]^{ab} (\partial_+ \hat{g} \hat{g}^{-1})_b. \quad (63)$$

K.Sfetsos and K.Siampos proved in [10] that for *Right* Poisson-Lie symmetric σ -models the quantum equivalence holds providing that the matrix M is invertible. In the Yang-Baxter σ -models this condition is always satisfied and the inverse of M is given by:

$$M^{-1} = \frac{\varepsilon^2}{1 + \varepsilon^2} \left(\frac{1}{\varepsilon} \mathbb{1} + R \right)$$

When we consider the dual model associated to the left action of G , the Drinfeld double is the cotangent bundle $T^*G = G \ltimes \mathcal{G}^*$. Then the dual group is the dual linear space \mathcal{G}^* of \mathcal{G} , which is an Abelian group with the addition of vectors as the group law. The corresponding action is that of the non-Abelian T -dual σ -models [6, 7, 8] and has the well-known expression:

$$\mathcal{S}(\hat{g} = e^{s\chi}) = \frac{1}{2\varepsilon t} \int d\xi^+ d\xi^- \partial_- \chi_a ((M^{-1})_{ab} + f_{ab}^c \chi_c)^{-1} \partial_+ \chi_b, \quad \chi \in \mathcal{G}^*, \quad s \in \mathbb{R}. \quad (64)$$

It has been showed in [19] that those models are one-loop renormalizable. Since the action (64) is *Left* Poisson-Lie symmetric, Sfetsos-Siampos condition [10] still holds (in their *Left* formulation) and implies again the quantum equivalence at one-loop.

7 Outlooks

Yang-Baxter σ -models are one case of non-trivial Poisson-Lie symmetric σ -models which keep the renormalizability and the quantum equivalence at the one-loop level, and are known to be classically integrable. Those models appear to be a semi-classical q -deformation of Poisson algebra, and can be a starting point in the quest for a quantum q -deformation fully renormalizable thanks to the relative simplicity of these models containing only two parameters .

Furthermore, for low dimensional compact Lie groups G the geometry associated to the Yang-Baxter σ -models can be viewed as a torsionless Einstein-Weyl geometry. We plan in the future to study the Weyl connections with torsion on Einstein manifolds, with the hope to learn more about the geometric aspects of the Poisson-Lie σ -models.

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References

- [1] C. Klimčík, *Yang-Baxter σ -models and dS/AdS T-duality*, JHEP 0212 (2002) 051, hep-th/0210095
- [2] C. Klimčík, *On integrability of the Yang-Baxter σ -model*, J.Math.Phys. 50:043508,2009, hep-th/08023518.
- [3] C. Klimčík and P. Ševera, *Dual Non-Abelian Duality and the Drinfeld Double*, Phys.Lett.B351 (1995) 455-462, hep-th/9502122
- [4] C. Klimčík and P. Ševera, *Poisson-Lie T-duality and Loop Groups of Drinfeld Doubles*, Phys.Lett. B372 (1996) 65-71, hep-th/9512040
- [5] K. Kikkawa, M. Yamasaki, Phys. Lett. B 149 (1984) 357;
N. Sakai, I. Senda, Prog. Theor. Phys. 75 (1986) 692.
- [6] X. de la Ossa and F. Quevedo, Nucl. Phys. B403 (1993) 377.
- [7] B.E. Fridling, A. Jevicki, Phys. Lett. B 134 (1984) 70.
- [8] E.S. Fradkin, A.A. Tseytlin, Ann. Phys. 162 (1985) 31.

- [9] V.G. Drinfeld, *Quantum Groups*, in Proc. ICM, MSRI, Berkeley (1986) p. 708;
F. Falceto, K. Gawędzki, J. Geom. Phys. 11 (1993) 251;
A.Yu. Alekseev, A.Z. Malkin, Commun. Math. Phys. 162 (1994) 147.
- [10] Konstadinos Sfetsos, Konstadinos Siampos *Quantum equivalence in Poisson-Lie T-duality*, JHEP 0906 (2009) 082, hep-th/0904.4248.
- [11] K. Sfetsos, Phys. Lett. B 432 (1998) 365.
- [12] G. Valent, C. Klimčík, R. Squellari, *One loop renormalizability of the Poisson-Lie σ -models*, Phys. Lett. B678 (2009) 143-148, hep-th/0902.1459.
- [13] K. Sfetsos, K. Siampos, Daniel C. Thompson, *Renormalization of Lorentz non-invariant actions and manifest T-duality*, Nucl.Phys.B827:545-564, 2010.
- [14] I. Kawaguchi and K. Yoshida, *Hybrid classical integrability in squashed sigma models*, Phys. Lett. B705 (2011) 251254, [arXiv:1107.3662].
- [15] I. Kawaguchi, T. Matsumoto, and K. Yoshida, *On the classical equivalence of monodromy matrices in squashed sigma model*, JHEP 1206 (2012) 082, [arXiv:1203.3400].
- [16] Io Kawaguchi, Takuya Matsumoto, Kentaroh Yoshida, *The classical origin of quantum affine algebra in squashed sigma models*, 10.1007/JHEP04(2012)115, hep-th/1201.3058
- [17] Francois Delduc, Marc Magro, Benoit Vicedo, *On classical q-deformations of integrable sigma-models*, hep-th/1308.3581. (2013)
- [18] B. E. Friedling and A. E. M. Van de Ven, Nucl. Phys. B 268 (1986) 719.
- [19] P. Y. Casteill, G. Valent *Quantum structure of T-dualized models with symmetry breaking*, Nucl.Phys. B591 (2000) 491-514, hep-th/0006186.
- [20] C.Klimčík, *Quasitriangular WZW model*, Rev. Math. Phys. 16 (2004) 679-808, hep-th/0103118.
- [21] Robert Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications*, ISBN 0-486-44529-1, 2006.
- [22] Dan Israël, Costas Kounnas, Domenico Orlando, and P. Marios Petropoulos *Heterotic strings on homogeneous spaces* Fortsch. Phys. 53 (2005) 1030-1071, hep-th/0412220.